An Integral Method Theorem for Heat Conduction

Robert L. Potts*
Science Applications, Inc., El Segundo, California

Abstract

A MATHEMATICAL solution existence and uniqueness theorem has been developed for applying the classical Goodman heat balance integral (HBI) method to a class of one-dimensional heat conduction problems. This Synoptic presents the theorem and points out its significance and uses. The theorem applies to the class of problems in which a heat flux load is imposed on a semi-infinite solid with temperature-dependent properties. It includes new mathematical conditions which test whether the HBI method can rigorously handle nonlinearities in the heat loads and material properties of a particular problem. Examples show that violation of the new conditions can lead to mathematical/numerical solutions which become unexpectedly singular under the HBI method, even though the original problem under study may be physically sound.

Contents

This research[1] grew out of a requirement that a certain large-scale computer program be guaranteed to run for any reasonable input. The program performed parametric calculations involving thousands of trial-and-error fly-downs of a user-specified re-entry vehicle (RV). A particular module of subroutines computed the ablation/erosion/heat conduction response of the RV heat shield. The Goodman heat balance integral method[2] was employed in the module and was found to provide the requisite levels of speed and accuracy. However, unexpected convergence problems occurred in the iterative solution of the method for some cases.

Analysis revealed that neither errors in programming nor implementation of numerical methods were at fault. Instead, the underlying HBI equations unexpectedly became mathematically invalid (singular) in certain physically reasonable domains. A mathematical existence and uniqueness proof was in fact required to define with absolute certainty an adequate domain of validity.[1] Questions of mathematical validity and applicability seem to have been much neglected in the literature on the HBI method, even though this example demonstrates their very practical importance.

Figure 1 depicts the physical situation. A semi-infinite slab with temperature-dependent properties is subjected to a net incoming heat flux $\dot{q}$. Initially, the slab is cold, and over time t it may recede by amount $s(t)$ due to ablation, erosion, or melting. The HBI method is used to calculate the surface (wall) temperature and recession histories.

Because of nonlinearities stemming from the temperature-dependent properties, it is convenient to transform the temperature T [K] to a new dependent variable, heat density H [J/cm^3]. (Units are provided in square brackets to clarify key variables.) A well-known transformation of Goodman is used.[2,3] Thus,

$$\theta = \int_{T_{\rm ref}}^{T_w} \rho c_p \mathrm{d}T \tag{1}$$

transforms T_w, the wall temperature, to θ, the heat density H at the wall surface ($z=0$). Here $T_{\rm ref}$ is a reference temperature at a position very deep inside the slab ($z\rightarrow\infty$). The specific heat c_p [J/g·K], density ρ [g/cm^3], thermal conductivity k [J/cm·s·K], and thermal diffusivity $\alpha = k/(\rho c_p)$ [cm^2/s] are considered known functions of temperature T for the slab of material.

For problems of interest, the heat flux $\dot{q}$ [J/cm^2·s] and surface recession rate $\dot{s}$ [cm/s] are considered known functions of time and wall temperature (equivalently, wall heat density); i.e.,

$$\dot{q} = \dot{q}(t,\theta) \tag{2}$$

$$\dot{s} = \dot{s}(t,\theta) \tag{3}$$

Thus the class of applicable problems includes situations with no recession ($\dot{s}\equiv 0$), as well as ablation modeled according to the modern, chemical kinetics viewpoint.[4,5] The older, more primitive view of ablation (i.e., melting of a solid at fixed wall temperature with complete removal of melt[2,3]) is not covered in this analysis except during the "preablation" period.

Under the classical HBI method, the original partial differential heat conduction equation is integrated once spatially, and a profile for the spatial heat distribution is assumed.[1-3] Using the front-face heat flux boundary condition, it is then possible to eliminate the thermal penetration depth $\delta(t)$ in favor of wall heat density $\theta(t)$ to yield an ordinary differential equation in $\theta(t)$ of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[K^*\theta^2\alpha/\dot{q}] = \dot{q} - \theta\dot{s} \tag{4}$$

for the problems of interest.[1] Equation (4) is known as a heat balance integral[2] and has the interpretation that the rate of change in heat storage equals the heat entering the slab surface minus the heat lost due to recession. Here K^* is the applicable dimensionless profile constant; for example, if the polynomial profile $H(z,t) = \theta(t)[1-z/\delta(t)]^n$ is used for the assumed heat distribution, then $K^* = n/(n+1)$, as shown in Ref. 1.

The goal is to solve Eq. (4) for $\theta(t)$, which in general must be done numerically. It does not seem to be well recognized, however, that singularities may be present in the problem other than the one already at $\theta=0$. For example, physically reasonable α variations may produce nonzero θ singularities in Eq. (4), causing the analytical solution to jump discontinuously and the numerical solution to fail (Fig. 2; see also Ref. 1, example 3). Even if no nonzero singularities are present in Eq. (4) per se, the solution may still become unbounded in a finite time span due to nonlinearities in $\dot{q}$ (see Ref. 1, example 4). To circumvent such difficulties, the following theorem was developed. (In the theorem below, Eq. (4) is re-expressed in standard canonical form for first-order ordinary differential equations.)

Theorem: Given the initial value problem [Eqs. (5) and (6)]

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \left[\dot{q} - \theta\dot{s} + K^*\frac{\theta^2\alpha}{\dot{q}^2}\frac{\partial \dot{q}}{\partial t}\right]\Bigg/\left[K^*\frac{\theta^2\alpha}{\dot{q}}\left\{\frac{2}{\theta} + \frac{1}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}\theta} - \frac{1}{\dot{q}}\frac{\partial\dot{q}}{\partial\theta}\right\}\right] \tag{5}$$

Presented as Paper 81-0283 at the AIAA 19th Aerospace Sciences Meeting, St. Louis, Mo., Jan. 12-15, 1981; submitted April 22, 1981; Synoptic received Aug. 23, 1982. Full paper available from AIAA Library, 555 West 57th Street, New York, N. Y. 10019. Price: Microfiche, $4.00; hard copy, $8.00. **Remittance must accompany order.**

*Staff Scientist. Member AIAA.

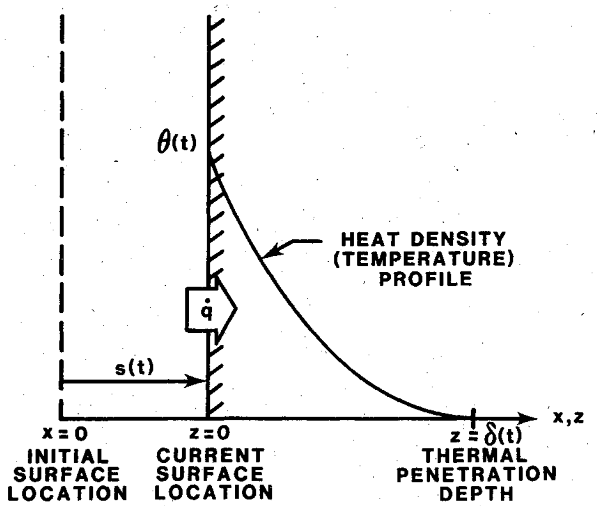

Fig. 1 Schematic of semi-infinite slab.

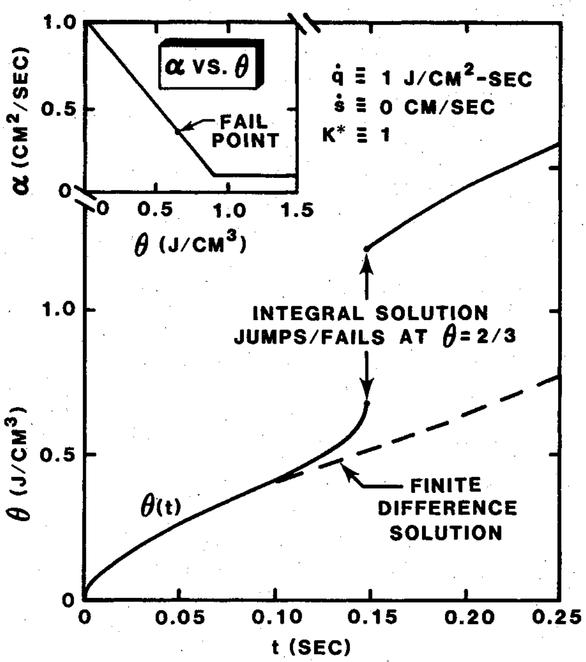

Fig. 2 Example of catastrophic failure of integral solution.

$$\theta(0) = \theta_0 > 0 \tag{6}$$

on the domain $S = \{(t,\theta): 0 \leq \theta \leq \infty, 0 \leq t \leq t_f < \infty\}$ with K^*, t_f (final time), and θ_0 positive constants, there exists a unique solution $\theta(t)$ to Eqs. (5) and (6) if the following conditions are satisfied:

1) $\alpha(\theta)$ is defined, strictly positive, bounded away from zero, and twice continuously differentiable on $\{\theta: 0 \leq \theta \leq \infty\}$.
2) $d[\theta^2\alpha(\theta)]/d\theta > 0$ for all $\theta > 0$.
3) $\dot{q}(t,\theta)$ is defined, strictly positive, and twice continuously differentiable on S.
4) $\partial\dot{q}/\partial\theta \leq 0$ everywhere on S.
5) $\dot{s}(t,\theta)$ is defined, nonnegative, and continuously differentiable on S.

The proof of the theorem is long and complicated and is given in Ref. 1. Per standard practice, the notions of continuity and differentiability on the boundary of the closed set S are defined with respect to the relative topology on S generated by restricting the usual topology of the entire real plane to the semi-infinite, closed strip S. In other words, on the $\theta = 0$ boundary, for example, the right derivative $\partial/\partial_R\theta$ is implied in place of the usual "two-sided" derivative $\partial/\partial\theta$ in the conditions above. Thus values at $\theta = 0$ for $d\alpha/d\theta$, $\partial\dot{q}/\partial\theta$, etc., are specifically being assumed to exist as finite numbers.

The requirement that $\dot{q}$ be strictly positive has been noted before,[2] and the other parts of conditions 1, 3, and 5 are generally tacitly assumed. Conditions 2 and 4 are believed to be new, however, and they have been specifically designed so that HBI practitioners can easily use them. Condition 2 has been used to determine whether a material with a given α function can be handled by the HBI method. For example, the $\alpha(\theta) = 1 - \theta$ function in Fig. 2 violates condition 2 when $\theta \geq 2/3$. Condition 2 has also been used to determine the maximum allowable spacing in a $\alpha - \theta$ data table in a HBI computer program for those materials whose physical α curves narrowly avoid violating condition 2. Overall, the theorem has been used to help guarantee reliability in deliverable software. Obviously, the theorem is not restricted to just ablation or re-entry problems, but can be applied to any situation that fits the mold of Eqs. (2) and (3). Additionally, it is believed that the method of proof in Ref. 1 can be applied to HBI problems involving boundary conditions other than those given by Eqs. (2) and (3).

Acknowledgments

This work was performed for the Ballistic Missile Office on behalf of the Strategic Air Command under U.S. Air Force Contract F04704-75-C-0022. The BMO Project Officer was Maj. Thomas Brandt, the SAI Program Manager was Dr. William H. Plows, and the SAI Task Manager was Bruce A. Bluestein.

References

[1] Potts, R. L., "Analysis of an Integral Method for Computing Reentry Vehicle Heat Conduction in Systems Level Computer Codes," AIAA Paper 81-0283, Jan. 1981.

[2] Goodman, T. R., "Application of Integral Methods to Transient Nonlinear Heat Transfer," *Advances in Heat Transfer*, Vol. I, edited by T. F. Irvine and J. P. Hartnett, Academic Press, New York, 1964, pp. 51-122.

[3] Özisik, M. N., *Boundary Value Problems of Heat Conduction*, International Textbook Company, Scranton, Pa., 1968, pp. 301-347.

[4] Mills, A. F., "Convective Heat and Mass Transfer to Re-entry Vehicles," School of Engineering and Applied Science, University of California, Los Angeles, Rept. UCLA-ENG-7892, Dec. 1978.

[5] Scala, S. M. and Gilbert, L. M., "Sublimation of Graphite at Hypersonic Speeds," *AIAA Journal*, Vol. 3, Sept. 1965, pp. 1635-1644.